

Packing designs with block size 6 and index 5

Ahmed M. Assaf

Department of Mathematics, Central Michigan University, Mt. Pleasant, MI 48859, USA

Alan Hartman

IBM-Science and Technology, Technion City, Haifa 32000, Israel

N. Shalaby

Department of Mathematics, University of Toronto, Toronto, Ont., Canada M1C 1A4

Received 16 January 1990

Revised 6 August 1990

Abstract

Assaf, A.M., A. Hartman and N. Shalaby, Packing designs with block size 6 and index 5, *Discrete Mathematics* 103 (1992) 121–128.

A (v, κ, λ) packing design of order v , block size κ and index λ is a collection of κ -element subsets, called blocks, of a v -set V such that every 2-subset of V occurs in at most λ blocks. The packing problem is to determine the maximum number of blocks in a packing design. The only previous work on the packing problem with $\kappa = 6$ concerns itself with the cases where the maximum packing design is in fact a balanced incomplete block design. In this paper we solve the packing problem with $\kappa = 6$ and $\lambda = 5$ and all positive integers v with the possible exceptions of $v = 41, 47, 53, 59, 62, 71$.

1. Introduction

A (v, κ, λ) packing design of order v , block size κ , and index λ is a collection, β , of κ -element subsets, called blocks, of a v -set, V , such that every 2-subset of V occurs in at most λ blocks.

Let $\sigma(v, \kappa, \lambda)$ denote the maximum number of blocks in a (v, κ, λ) packing design. A (v, κ, λ) packing design with $|\beta| = \sigma(v, \kappa, \lambda)$ will be called a maximum packing design. The function $\sigma(v, \kappa, \lambda)$ is of importance in coding theory since the block incidence vectors of a (v, κ, λ) packing design form the codewords of a binary code of length v minimum distance $2(\kappa - 1)$ and constant weight κ . Thus $\sigma(v, \kappa, \lambda)$ is the maximum number of codewords in such a code.

Schoenheim [8] has shown that

$$\sigma(v, \kappa, \lambda) \leq \left\lceil \frac{v}{\kappa} \left\lceil \frac{v-1}{\kappa-1} \lambda \right\rceil \right\rceil = \psi(v, \kappa, \lambda),$$

where $\lceil x \rceil$ is the largest integer satisfying $\lceil x \rceil \leq x$.

The value of $\sigma(v, 3, \lambda)$ for all v and λ has been determined by Schoenheim [8], and Hanani [5]. The value of $\sigma(v, 4, 1)$ has been determined for all v by Brouwer [4]; and the value of $\sigma(v, 4, \lambda)$ for all v and $\lambda > 1$ has been determined by Billington, Stanton and Stinson [3], Assaf [1], and Hartman [6].

In order to state the results known about $\sigma(v, 6, 5)$ we need the following definition. A balanced incomplete block design, $B[v, \kappa, \lambda]$ is a (v, κ, λ) packing design where every 2-subset of points is contained in precisely λ blocks. If a $B[v, \kappa, \lambda]$ exists, then it is clear that $\sigma(v, \kappa, \lambda) = \lambda v(v-1)/\kappa(\kappa-1) = \psi(v, \kappa, \lambda)$ and Hanani [5] proved the following existence theorem for $B[v, 6, \lambda]$.

Theorem 1.1. *Let $\lambda > 1$ and $v \geq 6$ be integers. Necessary and sufficient conditions for the existence of $B[v, 6, \lambda]$ are that $\lambda(v-1) \equiv 0 \pmod{5}$, and $\lambda v(v-1) \equiv 0 \pmod{30}$.*

Hanani and other authors (most notably Mills) have also shown that this condition is necessary and sufficient for the existence of $B[v, 6, 1]$ with the exception of $v = 36$ and a list of about 100 other possible exceptions (see [7] for a recent list).

This theorem implies that $\sigma(v, 6, 5) = \psi(v, 6, 5)$ for all $v \equiv 0, 1 \pmod{3}$. In this paper we are interested in determining the remaining values of $\sigma(v, 6, 5)$. Our goal is to prove that $\sigma(v, 6, 5) = \psi(v, 6, 5)$ for all v with some few possible exceptions. Specifically we prove the following.

Theorem 1.2. *For all positive integers $v \geq 6$ and for $v \leq 2$, we have $\sigma(v, 6, 5) = \psi(v, 6, 5)$ with the exception of $v = 8$ and possible exceptions of $v = 41, 47, 53, 59, 62, 71$.*

2. Recursive constructions of packing designs

In order to describe our recursive constructions we need the notions of designs with a hole, transversal designs and truncated transversal designs.

Let (V, β) be a (v, κ, λ) packing design, and let H be a subset of V of cardinality h . We shall say that (V, β) is an exact packing design with a hole of size h if no 2-subset of H appears in any block, and every other 2-subset of V appears in precisely λ blocks.

Lemma 2.1. *If $v \equiv 2 \pmod{3}$ then $\sigma(v, 6, 5) = \psi(v, 6, 5)$ if and only if there exists an exact $(v, 6, 5)$ packing with a hole of size 2.*

Proof. An easy computation shows that the number of blocks in an exact $(v, 6, 5)$ packing design with a hole of size 2 is $\psi(v, 6, 5)$. Conversely if a $(v, 6, 5)$ packing design exists with $\psi(v, 6, 5)$ blocks, then the number of pairs not covered is 5. Furthermore, the multi-graph of pairs not covered has every vertex of degree congruent to 0 mod 5. The only graph satisfying these requirements is one with $v - 2$ isolated vertices, and 2 vertices joined by 5 parallel edges. Hence the blocks of the design are an exact packing with a hole of size 2. \square

A necessary condition for the existence of an exact (v, κ, λ) packing with a hole of size h is given by the following.

Lemma 2.2. *If there exists an exact (v, κ, λ) packing design with a hole of size h then $v \geq (\kappa - 1)h + 1$.*

Proof. The number of blocks containing a point of the hole is $\lambda(v - h)/(\kappa - 1)$. Since no block contains two points of the hole, the total number of blocks in the design containing some point of the hole is therefore $\lambda h(v - h)/(\kappa - 1)$.

Now these blocks between them contain $\binom{\kappa-1}{2} \lambda h(v - h)/(\kappa - 1)$ pairs of points—neither of which is in the hole. However the total number of such pairs covered by blocks of the design is $\lambda \binom{v-h}{2}$, and hence

$$\binom{\kappa-1}{2} \frac{\lambda h(v-h)}{\kappa-1} \leq \lambda \binom{v-h}{2},$$

which implies the result. \square

Let κ, λ, m and v be positive integers. A group divisible design $\text{GD}[\kappa, \lambda, m, v]$ is a triple (V, β, γ) where V is a set of points with $|V| = v$, and $\gamma = \{G_1, \dots, G_l\}$ is a partition of V into l sets of size m . The parts G_i of the partition are called groups. The collection β consists of κ -subsets of V , called blocks with the following properties:

- (1) $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$;
- (2) every 2-subset $\{x, y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

A group divisible design $\text{GD}[\kappa, \lambda, w, \kappa w]$ is called a transversal design denoted by $T[\kappa, \lambda, w]$. It is well known that a $T[\kappa, 1, w]$ is equivalent to $\kappa - 2$ mutually orthogonal Latin squares of side w . The following existence theorems for transversal designs are most useful for us. The proofs of these results may be found in [2].

Theorem 2.1. *There exists a $T[7, 1, w]$ for all positive integers w with the exception of $w \in \{2, 3, 4, 5, 6\}$ and possible exception of $w \in \{10, 14, 15, 18, 20, 21, 22, 24, 26, 28, 30, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 51, 52, 54, 60, 62\}$. Furthermore, there exists a $T[10, 1, w]$ for $w \in \{11, 23\}$.*

Theorem 2.2. *There exists a $T[7, \lambda, w]$ for all positive integers w and all integers $\lambda \geq 2$.*

We now give the definition of truncated transversal design. Let κ , λ and w be positive integers, and let u be nonnegative integer. A truncated transversal design $TT[\kappa, \lambda, w, u]$ is a triple (V, β, γ) where V is a set of points with $|V| = (\kappa - 1)w + u$, and $\gamma = \{G_1, \dots, G_\kappa\}$ is a partition of V into $\kappa - 1$ sets of size w and one set G_κ of size u . G_i are called the groups of the truncated transversal design. The collection β consists of κ -subsets and $(\kappa - 1)$ -subsets of V , called blocks with the following properties

- (1) $|B \cap G_i| = 1$ for all $B \in \beta$ and $1 \leq i < \kappa$;
- (2) $|B \cap G_\kappa| = 1$ for all $B \in \beta$ such that $|B| = \kappa$;
- (3) every 2-subset $\{x, y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

Clearly $TT[\kappa, \lambda, w, 0]$ is equivalent to a $T[\kappa - 1, \lambda, w]$. Furthermore, if $0 \leq u \leq w$ then one may construct a $TT[\kappa, \lambda, w, u]$ from a $T[\kappa, \lambda, w]$ by removing points from the last group, and from all the blocks which contain them. Thus we have the following existence results which are in the form most useful to us.

Theorem 2.3. *There exists a $TT[7, 1, w, u]$ for all integers $0 \leq u \leq w$ and for all positive integers w with the exception of $w \in \{2, 3, 4, 5, 6\}$ and the same possible exception as in Theorem 2.1. Furthermore there exists designs $TT[10, 1, 11, 1]$, $TT[10, 1, 23, 22]$ and $TT[10, 1, 23, 19]$.*

Theorem 2.4. *There exists a $T[6, 5, w]$ for all positive integers w .*

We can now give the recursive constructions used in the proof of our main theorem.

Theorem 2.5. *If there exists a $TT[3n + 1, 1, w, u]$ with $n \leq 2$, $w = 0$ or $1 \pmod{3}$, $w \geq 6$, and $\sigma(u, 6, 5) = \psi(u, 6, 5)$ then $\sigma(3nw + u, 6, 5) = \psi(3nw + u, 6, 5)$.*

Proof. On the blocks and the groups of size w of the truncated transversal design construct balanced incomplete block designs $B[v, 6, 5]$ with $v = 3n$, $3n + 1$, and w . On the group of size u construct a $(u, 6, 5)$ packing design with $\psi(u, 6, 5)$ blocks. This gives us a $(3nw + u, 6, 5)$ packing design with $\psi(3nw + u, 6, 5)$ blocks. \square

Let us now add h new points to the points of a $TT[3n + 1, 1, w, u]$ with $n \geq 2$. On each block construct a $B[v, 6, 5]$ with $v = 3n$ or $3n + 1$. For each group of size w construct a $(w + h, 6, 5)$ packing design with a hole on the h new points. Now construct a $(u + h, 6, 5)$ packing design on the last group and the new points. If these last two designs exist with the maximum possible number of blocks, then

the resulting design is a $(3nw + u + h, 6, 5)$ packing design with $\psi(3nw + u + h, 6, 5)$ blocks. This construction proves the following generalization of Theorem 2.5.

Theorem 2.6. *If there exists a $TT[3n + 1, 1, w, u]$ with $n \geq 2$, and there exists an exact $(w + h, 6, 5)$ packing design with a hole of size h , and $\sigma(u + h, 6, 5) = \psi(u + h, 6, 5)$, then $\sigma(3nw + u + h, 6, 5) = \psi(3nw + u + h, 6, 5)$.*

Setting $h = 0$ gives us Theorem 2.5. However, Theorem 2.6 can be used with $h = 1$ and $w \equiv 0 \pmod{3}$, in which case the hole is trivial, and also when $h = 2$ since the maximum $(w + 2, 6, 5)$ packing designs are exact designs with a hole of size 2.

A similar argument also proves the following.

Theorem 2.7. *If there exists a $T[6, 5, w]$ and there exists an exact $(w + h, 6, 5)$ packing design with a hole of size h , and $\sigma(w + h, 6, 5) = \psi(w + h, 6, 5)$, then $\sigma(6w + h, 6, 5) = \psi(6w + h, 6, 5)$.*

The following theorems are used to construct packing on $6w + h$ points even when we are unsure of the existence of a $T[7, 1, w]$. In particular when $w = 14, 18, 21$ and 22 we use the result below to construct optimal packing.

Theorem 2.8. *If there exists a $TT[7, 1, w, u]$ then there exists a $GD[6, 5, \{2w, (2u)^*\}, 12w + 2u]$ where $*$ means that there is exactly one group of size $2u$.*

Proof. Let X be the pointset of a $TT[7, 1, w, u]$ and construct a $GD[6, 5, \{2w, (2u)^*\}, 12w + 2u]$ by replacing each point $x \in X$ by two points $\{x_0, x_1\}$ so the groups are of size $2w$ and $2u$. On each block B of size 6 construct a $GD[6, 5, 2, 12]$ in such a way that it has groups $\{b_0, b_1\}$ for $b \in B$. Such design exists by Theorem 2.4; and on each block of size 7 construct a $GD[6, 5, 2, 14]$ where the groups are $\{b_0, b_1\}$ for $b \in B$. A $GD[6, 5, 2, 14]$ can be constructed as follows. Let $X = \mathbb{Z}_2 \times \mathbb{Z}_7$, then the required blocks are

$$\begin{aligned} &\langle (0, 0) (0, 1) (0, 2) (0, 3) (1, 5) (1, 6) \rangle \text{ mod } (-, 7), \\ &\langle (0, 0) (0, 2) (0, 4) (0, 5) (1, 1) (1, 6) \rangle \text{ mod } (-, 7), \\ &\langle (0, 0) (0, 1) (0, 4) (1, 2) (1, 3) (1, 5) \rangle \text{ mod } (-, 7), \\ &\langle (0, 0) (1, 1) (1, 2) (1, 3) (1, 5) (1, 6) \rangle \text{ mod } (-, 7). \quad \square \end{aligned}$$

Similarly if in the previous theorem we replace each point $x \in X$ by 3 points $\{x_0, x_1, x_2\}$ then we have the following.

Theorem 2.9. *If there exists a $TT[7, 1, w, u]$ then there exists a $GD[6, 5, \{3w, (3u)^*\}, 18w + 3u]$.*

Proof. The proof of this theorem is the same as Theorem 2.8. We only need to prove that there exists a $\text{GD}[6, 5, 3, 21]$. Let $X = \mathbb{Z}_{21}$ and let the groups be $G_i = \{i, i + 7, i + 14\}$, $i = 0, 1, \dots, 6$. The blocks are

$$\begin{aligned} &\langle 0 \ 1 \ 2 \ 4 \ 12 \ 17 \rangle \bmod 21, \\ &\langle 0 \ 1 \ 4 \ 6 \ 9 \ 10 \rangle \bmod 21, \\ &\langle 0 \ 2 \ 8 \ 12 \ 18 \ 20 \rangle \bmod 21. \quad \square \end{aligned}$$

Apply the argument of Theorem 2.6 to Theorems 2.8 and 2.9 we get the following recursive construction.

Theorem 2.10. *If there exists a $\text{GD}[6, 5, \{w, u^*\}, 6w + u]$ and there exists a $(w + h, 6, 5)$ packing design of a hole of size h , and there exists a $(u + h, 6, 5)$ packing design with $\psi(u + h, 6, 5)$ blocks, then $\sigma(6w + u + h, 6, 5) = \psi(6w + u + h, 6, 5)$.*

3. The main theorem

Before giving an induction proof of Theorem 1.2, we need the following constructions of packing design with small values of v .

Lemma 3.1. $\sigma(v, 6, 5) = \psi(v, 6, 5)$ for all $v = 11, 14, 17, 20, 23, 26, 29, 32, 35, 38$.

Proof. The constructions of these packing designs are given in Table 1. In general the construction is as follows. Let $X = \mathbb{Z}_n \cup \{a, b\}$. The blocks are constructed by taking the orbits of the tabulated base blocks under the action of the cyclic group generated by the permutation which fixes the elements $\{a, b\}$ and sends $i \rightarrow i + 1 \pmod{n}$ for each $i \in \mathbb{Z}_n$. \square

Lemma 3.2. $\sigma(8, 6, 5) = \psi(8, 6, 5) - 1 = 8$.

Proof. By Lemmas 2.1 and 2.2, we have that $\sigma(8, 6, 5) < \psi(8, 6, 5)$. To prove the lemma, it suffices to exhibit an $(8, 6, 5)$ packing with 8 blocks. Let $X = \mathbb{Z}_8$ and take the blocks $\langle 0, 1, 3, 4, 6, 7 \rangle \bmod 8$. \square

We are now able to prove our main theorem, which is restated below for the reader's convenience.

Theorem 1.2. *For all positive integers $v \geq 6$, and for $v \leq 2$, we have $\sigma(v, 6, 5) = \psi(v, 6, 5)$ with the exception of $v = 8$ and the possible exceptions of $v = 41, 47, 53, 59, 62, 71$.*

Table 1
Packing designs with block size 6 and index 5

v	Point set	Block set	
11	$\mathbb{Z}_9 \cup \{a, b\}$	$\langle 0, 1, 3, 4, 5, a \rangle$ $\langle 0, 2, 3, 4, 6, b \rangle$	
14	$\mathbb{Z}_{12} \cup \{a, b\}$	$\langle 0, 1, 3, 6, 8, a \rangle$ $\langle 0, 1, 2, 3, 6, b \rangle$ $\langle 0, 2, 4, 6, 8, 10 \rangle$ $\langle 0, 1, 4, 5, 8, 9 \rangle$	(orbit length 2) (orbit length 4)
17	$\mathbb{Z}_{15} \cup \{a, b\}$	$\langle 0, 1, 3, 7, 13, a \rangle$ $\langle 0, 1, 3, 8, 14, b \rangle$ $\langle 0, 1, 3, 6, 7, 11 \rangle$	
20	$\mathbb{Z}_{18} \cup \{a, b\}$	$\langle 0, 1, 2, 4, 9, a \rangle$ $\langle 0, 1, 5, 9, 11, b \rangle$ $\langle 0, 3, 6, 9, 12, 15 \rangle$ $\langle 0, 1, 6, 7, 12, 13, \rangle$ $\langle 0, 1, 3, 5, 8, 15 \rangle$	(orbit length 3) (orbit length 6)
23	$\mathbb{Z}_{21} \cup \{a, b\}$	$\langle 0, 5, 8, 10, 12, a \rangle$ $\langle 1, 2, 10, 11, 16, b \rangle$ $\langle 1, 8, 14, 17, 18, 20 \rangle$ $\langle 9, 10, 12, 16, 17, 19 \rangle$	
26	$\mathbb{Z}_{24} \cup \{a, b\}$	$\langle 0, 1, 2, 4, 14, a \rangle$ $\langle 0, 1, 3, 9, 20, b \rangle$ $\langle 0, 4, 8, 12, 16, 20 \rangle$ $\langle 0, 1, 8, 9, 16, 17 \rangle$ $\langle 0, 1, 3, 6, 12, 19 \rangle$ $\langle 0, 2, 5, 9, 15, 19 \rangle$	(orbit length 4) (orbit length 8)
29	$\mathbb{Z}_{27} \cup \{a, b\}$	$\langle 0, 4, 7, 12, 13, a \rangle$ $\langle 0, 5, 9, 19, 21, b \rangle$ $\langle 1, 3, 8, 12, 13, 14 \rangle$ $\langle 0, 1, 2, 4, 9, 17 \rangle$ $\langle 1, 4, 7, 11, 14, 22 \rangle$	
32	$\mathbb{Z}_{30} \cup \{a, b\}$	$\langle 0, 1, 6, 22, 24, a \rangle$ $\langle 0, 3, 7, 11, 25, b \rangle$ $\langle 0, 5, 10, 15, 20, 25 \rangle$ $\langle 0, 1, 10, 11, 20, 21 \rangle$ $\langle 0, 1, 3, 8, 20, 24 \rangle$ $\langle 0, 1, 4, 10, 12, 17 \rangle$ $\langle 0, 1, 3, 16, 18, 27 \rangle$	(orbit length 5) (orbit length 10)
35	$\mathbb{Z}_{33} \cup \{a, b\}$	$\langle 0, 4, 7, 19, 27, a \rangle$ $\langle 0, 7, 11, 13, 21, b \rangle$ $\langle 1, 4, 9, 17, 20, 22 \rangle$ $\langle 0, 7, 11, 16, 17, 26 \rangle$ $\langle 0, 1, 2, 3, 7, 12 \rangle$ $\langle 0, 1, 3, 9, 18, 22 \rangle$	
38	$\mathbb{Z}_{36} \cup \{a, b\}$	$\langle 0, 1, 6, 14, 26, a \rangle$ $\langle 0, 2, 6, 9, 24, b \rangle$ $\langle 0, 1, 3, 18, 19, 21 \rangle$ $\langle 0, 1, 3, 10, 14, 31 \rangle$ $\langle 0, 1, 4, 10, 12, 17 \rangle$ $\langle 0, 1, 5, 12, 22, 28 \rangle$ $\langle 0, 2, 10, 13, 17, 22 \rangle$	(orbit length 18)

Proof. For $v \equiv 0$ or $1 \pmod{3}$ there exists a $B[v, 6, 5]$, so it only remains to consider values of $v \equiv 2 \pmod{3}$. For $v \leq 38$ the result is given by Lemmas 3.1 and 3.2. For $v \geq 41$, a certain amount of hand calculation shows that if $v \notin \{41, 47, 53, 59, 62, 65, 71, 86, 92, 95, 101, 122, 134, 137, 143, 146, 227, 230\}$ then v can be written in the form $v = 6w + u + h$, where w , u , and h are chosen so that

- (a) there exists a $TT[7, 1, w, u]$ (by Theorem 2.3);
- (b) $u + h \in \{2, 11, 14, 17, 20, 23, 26, 29, 32\}$;
- (c) if $w \equiv 2 \pmod{3}$ then $h = 1$ otherwise $h = 0$.

Now apply Theorem 2.6, and the result follows.

For $v \in \{65, 86\}$ apply Theorem 2.6 with $n = h = 2$, and $v = w = 9$ or 12 .

For $v = 92$ apply Theorem 2.7 with $w = 15$ and $h = 2$.

For $v \in \{95, 122, 134\}$ apply Theorem 2.8 to the designs $TT[7, 1, 7, 5]$, $TT[7, 1, 9, 7]$, and $TT[7, 1, 11, 1]$, then apply Theorem 2.10 with $h = 1$ for $v = 95$, and otherwise $h = 0$.

For $v \in \{101, 227, 230\}$ apply Theorem 2.6 with $n = 3$, $h = 1$, $w = 11$ or 23 , and $u = 1, 19$, or 22 .

For $v \in \{137, 143, 146\}$ apply Theorem 2.9 to the designs $TT[7, 1, 7, u]$ with $u = 3, 5, 6$, then apply Theorem 2.10 with $h = 2$.

This completes the proof. \square

Acknowledgement

N. Shalaby would like to thank A. Rosa for his helpful comments and encouragement.

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